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Quantum gates on hybrid qudits

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Abstract
We introduce quantum hybrid gates that act on qudits of different dimensions. In particular, we develop two representative two-qudit hybrid gates (SUM and SWAP) and many-qudit hybrid Toffoli and Fredkin gates. We apply the hybrid SUM gate to generating entanglement, and find that operator entanglement of the SUM gate is equal to the entanglement generated by it for certain initial states. We also show that the hybrid SUM gate acts as an automorphism on the Pauli group for two qudits of different dimensions under certain conditions. Finally, we describe a physical realization of these hybrid gates for spin systems.

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1. Introduction

Although quantum computation is treated as processing qubits (quantum versions of binary digits, or bits), quantum computing can be generalized by considering logical elements of qudits (quantum versions of d-ary digits) [1]. Qubit-based quantum computation is adequate for considering fundamental issues such as complexity classes or computability, but, from a practical perspective, encoding as qudits may be more natural, or constitute a more efficient use of resources [2]. For example, coupled harmonic oscillators can admit various qudit encodings that exploit the full Hilbert space [2, 3].

Two-qudit gates have been treated, but so far always for two qudits of equal dimensions [1]. Here we treat hybrid qudit gates, namely gates that transform two (or more) qudits of possibly different dimensions. This analysis is particularly useful if two or more qudits of different physical systems (and different dimensions) are coupled together (such as a $d = 2$ level system and a large d-dimensional qudit in an oscillator). We develop two- and multi-qudit hybrid gates, discuss possible physical realizations and prove that the hybrid SUM gate acts on the Pauli group for two qudits as an automorphism only when certain conditions on the dimensions of the qudit Hilbert spaces are met.

A *qudit* is a general state in a d-dimensional Hilbert space $\mathcal{H}_d$, i.e. $|\Psi\rangle = \sum_{m=0}^{d-1} c_m|m\rangle$, which reduces to $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ for the qubit case. A *basis* for a general multi-qudit system is given by

$$|m_1\rangle \otimes |m_2\rangle \otimes \cdots \otimes |m_N\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \cdots \otimes \mathcal{H}_{d_N} \qquad m_i \in \mathbb{Z}_{d_i}. \tag{1}$$

If two or more d_i differ, we refer to the multi-qudit system (1) as *'hybrid system'*.

This generalization is illuminating because it differs subtly from standard non-hybrid qudit models (see, e.g., lemma 2 in section 5). Moreover, hybrid systems have a wider range of applications. For example, a qubit can serve as a control state with any qudit as the target state, or vice versa. Also qubits are often only ideals: many systems involve multiple levels for each degree of freedom, and the qubit is encoded into these levels. The theory for hybrid qudit systems can be useful for different interacting physical systems, with a d_1-dimensional qudit natural for one system and a d_2-dimensional qudit natural for another.

This paper is organized as follows. In section 2 we first review the qudit computational basis and one-qudit operators. Then we construct two hybrid versions of the SUM gate (see equations (9) and (14) [2–4], a partial-SWAP gate and a hybrid version of the Toffoli [5–9] and Fredkin gates [10–14] that were instrumental in introducing the field of reversible (classical) computation. In section 3 we calculate the operator entanglement of the SUM gate and the entanglement generated by the SUM gate. In section 4 we describe a realization of the hybrid gates by spin systems. In section 5 we prove a lemma that shows the SUM gate yields an automorphism of the Pauli group by conjugation, if and only if the dimension of the control system is a multiple of that of the target system. We conclude in section 6.

2. Hybrid quantum gates

2.1. Generalized Pauli group

A basis for operators on $\mathcal{H}_d$ is given by the following 'generalized Pauli operators' [2, 3, 15, 16]:

$$X^j Z^k \qquad j, k \in \mathbb{Z}_d \tag{2}$$

where X and Z are defined by their action on the computational basis

$$X|s\rangle = |s + 1 (\mathrm{mod}\ d)\rangle \tag{3}$$

$$Z|s\rangle = \exp(2\pi \mathrm{i} s/d)|s\rangle = \zeta_d^s|s\rangle \tag{4}$$

where

$$\zeta_d \equiv \exp(\mathrm{i}2\pi/d). \tag{5}$$

In the following we shall write for simplicity ζ instead of ζ_d, if the dimension is easily understood from the context.

The unitary operators X and Z generate the *generalized Pauli group* $\mathcal{P}_d$. Note that X and Z do not commute; they obey

$$Z^j X^k = \zeta^{jk} X^k Z^j \tag{6}$$

and $X^d = Z^d = I$.

2.2. One-qudit gates

Before we consider two-qudit gates, we review some of the properties of the useful one-qudit 'Fourier gate' F, which transfers the qudit computational basis $|s\rangle$ to the dual state

$$|s) \equiv F|s\rangle := \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}\zeta^{sk}|k\rangle \qquad \text{for} \quad s \in \mathbb{Z}_d \tag{7}$$

such that $\langle s'|s) = 1/\sqrt{d}\zeta^{ss'}$. These dual states are related to the computational basis by a discrete Fourier transformation, and distinguished by a rounded bra/ket notation. As an example, if the computational basis corresponds to Fock number states for the harmonic oscillator, the dual basis corresponds to Susskind–Glogower phase states [17]. Similarly, the $SU(2)$ phase states are dual to angular momentum eigenstates [18].

The F gate is a qudit version of the one-qubit Hadamard gate H. However, and in contrast to H, the F operator for $d \geqslant 3$ is not Hermitian and its order is 4 instead of 2, as [19]

$$F^2|s\rangle = |-s\rangle \qquad F^4 = I. \tag{8}$$

Similarly, the unitary operator X can be considered as the qudit version of the NOT gate, and Z is the qudit version of the phase gate for qubits.

2.3. Two-qudit gates

2.3.1. Hybrid SUM gate. Two representative quantum gates on qubits are the controlled-NOT (CNOT) and SWAP gate. A generalized CNOT gate for qudits [2, 3, 20] has been called the displacement gate, or SUM gate [20]. As a compromise, we refer to the hybrid version of this 'controlled-SHIFT' operator as the 'SUM gate', but use the notation $\mathcal{D}$ to emphasize its displacement nature. To achieve unity in the notation, we shall use calligraphic letters to denote two- and multi-qudit gates. In particular, we shall use $\mathcal{S}$, $\mathcal{T}$ and $\mathcal{F}$ to denote the SWAP, the hybrid Toffoli and Fredkin gates, respectively.

We now define the hybrid version of the SUM or displacement gate $\mathcal{D}$ on $\mathcal{H}_{d_c} \otimes \mathcal{H}_{d_t}$ for arbitrary d_c and d_t (the subscript c refers to 'control' and t to 'target') by

$$\mathcal{D} := \sum_{n=0}^{d_c-1} P_n \otimes X^n \qquad \text{for} \quad d_c, d_t \in \mathbb{N} \tag{9}$$

where

$$P_n \equiv |n\rangle\langle n| \qquad n \in \mathbb{Z}_{d_c} \tag{10}$$

is a *primitive projection operator* on a computational basis state of the *control space* $\mathcal{H}_{d_c}$.

It is important to note the following subtle difference between hybrid and non-hybrid qudit systems: although the states $|i\rangle \otimes |j\rangle$ and $|i + d_c\rangle \otimes |j\rangle$ are formally equivalent, the operators $P_i \otimes X^i = |i\rangle\langle i| \otimes X^i$ and $P_{i+d_c} \otimes X^{i+d_c} = |i + d_c\rangle\langle i + d_c| \otimes X^{i+d_c} = P_i \otimes X^{i+d_c}$ are *not* equal in general, if $d_c \neq d_t$. Hence, in order to obtain a unique definition, we insist that the summation in (9) is restricted to $0 \leqslant n < d_c$. This subtle difference has interesting consequences when we try to define a SWAP gate for hybrid systems.

We can combine together all the projection operators P_n, which yield the same X^s, and obtain

$$\mathcal{D} = \sum_{s=0}^{d_t-1} \Pi_s \otimes X^s \qquad \text{for} \quad d_c > d_t \tag{11}$$

where

$$\Pi_s = \sum_{n=s \bmod d_c}^{d_c-1} P_n \qquad \text{for} \quad s \in \mathbb{Z}_{d_t}. \tag{12}$$

For example, the SUM gate for $d_c = 3$ and $d_t = 2$ is given by

$$\mathcal{D} = \sum_{s=0}^{1} \Pi_s \otimes X^s = \Pi_0 \otimes I + \Pi_1 \otimes X$$

where $\Pi_0 = P_0 + P_2$ and $\Pi_1 = P_1$.

We can extend expression (11), also for $d_c \leqslant d_t$, by defining

$$\mathcal{D} := \sum_{s=0}^{d_{\min}-1} \Pi_s \otimes X^s \tag{13}$$

where $d_{\min} := \min(d_c, d_t)$. Note that $\sum_{s=0}^{d_{\min}-1} \Pi_s = I_{d_c \times d_c}$.

We introduce another interesting hybrid gate:

$$\mathcal{D}'_{12}|m\rangle \otimes |n\rangle := |m\rangle \otimes |m - n\rangle \qquad \text{for} \quad m \in \mathbb{Z}_{d_c} \quad \text{and} \quad n \in \mathbb{Z}_{d_t}. \tag{14}$$

This operator is unitary and Hermitian, as $(\mathcal{D}'_{12})^2 = I$. It is related to the SUM gate by

$$\mathcal{D}'_{12} = \mathcal{D}_{12}(I \otimes F^2).$$

For $d_c = d_t$ our hybrid $\mathcal{D}'_{12}$ reduces to the generalized CNOT gate given by Alber *et al* [4].

2.3.2. The SWAP gate. The SWAP operation on $\mathcal{H}_d \times \mathcal{H}_d$ systems, i.e. for $d_c = d_t = d$ systems, is defined by

$$\mathcal{S}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle \qquad \text{for} \quad i, j \in \mathbb{Z}_d \tag{15}$$

hence, $\mathcal{S} = \sum_{i,j=0}^{d-1} |j\rangle\langle i| \otimes |i\rangle\langle j|$. Clearly, the definition cannot be used for hybrid systems. Instead, for $d_c \neq d_t$ (and also for $d_c = d_t$) we define partial-SWAP operators by

$$\mathcal{S}_P|i\rangle \otimes |j\rangle = \begin{cases} |j\rangle \otimes |i\rangle & \text{for} \quad i, j \in \mathbb{Z}_{d_P} \\ |i\rangle \otimes |j\rangle & \text{otherwise} \end{cases} \tag{16}$$

where $d_P \leqslant d_{\min} = \min(d_c, d_t)$. Obviously, $\mathcal{S}_P$ in (16) is unitary and Hermitian, as $\mathcal{S}_P^2 = I$. This partial SWAP gate only acts as a SWAP operation on a subspace of the original Hilbert space.

2.3.3. Relation between SWAP and SUM operators. It is easy to check that $\mathcal{S}$ can be written in terms of three SUM gates as follows:

$$\mathcal{S} = (F^2 \otimes I)\mathcal{D}_{12}\mathcal{D}_{21}^{-1}\mathcal{D}_{12}. \tag{17}$$

Another possibility is to use expressions (17) formally to define a swap-like gate for hybrid system. However, contrary to what one might expect, this operator does not yield a swap operation, even for $0 \leqslant i, j \leqslant d_{\min}$.

We illustrate this claim by a simple example, where $d_1 = 3$ and $d_2 = 2$. By applying expression (17) to the state $|0\rangle \otimes |1\rangle$, we obtain successively

$$\begin{aligned} &|0\rangle \otimes |1\rangle \longrightarrow |0\rangle \otimes |1\rangle \longrightarrow |2\rangle \otimes |1\rangle \\ &\longrightarrow |2\rangle \otimes |1\rangle \longrightarrow |1\rangle \otimes |1\rangle \neq |1\rangle \otimes |0\rangle. \end{aligned} \tag{18}$$

Recently, Fujii constructed a swap gate as follows [21]:

$$\mathcal{S} = \mathcal{D}_{12}(F^2 \otimes I)\mathcal{D}_{21}(F^2 \otimes I)\mathcal{D}_{12}(I \otimes F^2) \tag{19}$$

expressed in our notation. Note that both constructions of SWAP gates actually require three SUM gates and three local F^2 gates. This is because

$$\mathcal{D}_{21}^{-1} = (I \otimes F^2)\mathcal{D}_{21}(I \otimes F^2) \tag{20}$$

so that our SWAP gate (17) can be written as

$$\mathcal{S} = (F^2 \otimes I)\mathcal{D}_{12}(I \otimes F^2)\mathcal{D}_{21}(I \otimes F^2)\mathcal{D}_{12}. \tag{21}$$

We also note that the SWAP gate on continuous variables can be constructed by three generalized controlled-NOT gates on continuous variables [22].

2.4. Higher-order quantum hybrid gates

Representative higher-order three-qubit gates include the quantum versions of the Toffoli gate [5–9] and of the Fredkin gate [10–14]; these three-bit gates are important primitives for logically reversible classical computation, for which universal reversible two-bit gates do not exist. The Toffoli gate is effectively a controlled-controlled-NOT (C^2 NOT), and the Fredkin gate is another universal three-bit gate.

As a controlled-controlled-NOT, the quantum Toffoli gate has two qubits as control and one qubit as target, and the target qubit flips if and only if the two control qubits are in the state $|1\rangle \otimes |1\rangle$. The Fredkin gate has one qubit as control and two qubits as target, and the states of two target qubits swap if and only if the control qubit is in the state $|1\rangle$. Here we give the hybrid version of these two higher-order gates.

2.4.1. The hybrid Toffoli gate. A general controlled unitary gate acting on Hilbert spaces $\mathcal{H}_{d_c} \otimes \mathcal{H}_{d_t}$ can be written as

$$\mathcal{C}_U = \sum_{s=0}^{d_c-1} P_s \otimes U_s = \sum_{s=0}^{d_c-1} |s\rangle\langle s| \otimes U_s \tag{22}$$

where U_s are arbitrary unitary operators on the *target space* $\mathcal{H}_{d_t}$.

Note that $\{U_s\}$ may be unitary operators on single or multiple qudits, and may include the case of qudit-controlled operators on other qudits. The latter case allows unitary operators on qudits that can be jointly controlled by two or more qudits. An example is provided by the following 'natural' generalization of the Toffoli gate [5–9]:

$$\mathcal{T} := \sum_{s=0}^{d_c-1} P_s \otimes \mathcal{D}^s \tag{23}$$

where the U_s in (22) are replaced by $\mathcal{D}^s$, which are powers of the generalized displacement operator (9). The hybrid Toffoli-type gate is thus a *'triple gate'*

$$\mathcal{T} = \sum_{r=0}^{d_c-1} \sum_{s=0}^{d_c'-1} P_r \otimes P_s \otimes X^{rs} = \sum_{m=0}^{d_t-1} \Pi_m \otimes X^m \tag{24}$$

where Π_m are compound projection operators, given by

$$\Pi_m = \sum_{r=0}^{d_c-1} \sum_{s=0}^{d_c'-1} \delta_{m,rs} P_r \otimes P_s \qquad m \in \mathbb{Z}_{d_t} \tag{25}$$

where the products rs of the delta in (25) are defined modulo d_t. Hence, the order of the Toffoli gate is equal to d_t.

2.4.2. The hybrid Fredkin gate. Another type of multi-qudit gate is the quantum Fredkin gate [10–14]. We define the *hybrid Fredkin gate* on $\mathcal{H}_{d_c} \otimes \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ by

$$\mathcal{F} := \sum_{m=0}^{d_c-1} P_m \otimes \mathcal{S}_P^m = \Pi_+ \otimes I + \Pi_- \otimes \mathcal{S}_P \tag{26}$$

where $\Pi_\pm$ are the following projection operators:

$$\Pi_+ := \sum_{\text{even}\, m} P_m \qquad \text{and} \qquad \Pi_- := \sum_{\text{odd}\, m} P_m \tag{27}$$

where we have used the property $\mathcal{S}_P^2 = I$.

The hybrid Fredkin gate executes a swap for purely odd state $|\psi_-\rangle$, i.e. for $\Pi_-|\psi_-\rangle = |\psi_-\rangle$, and does nothing for the even states. However, for mixed odd and even states, one obtains a mixed result. For instance, if we choose a input state as $(|0\rangle + |1\rangle) \otimes |\alpha\rangle \otimes |\beta\rangle$, the output state after the gate is $|0\rangle \otimes |\alpha\rangle \otimes |\beta\rangle + |1\rangle \otimes |\beta\rangle \otimes |\alpha\rangle$, which is in general an entangled state.

3. Entanglement produced by quantum gates

Hybrid two- and multi-qudit gates can enhance entanglement, i.e. the entanglement of the output state can be greater than that of the input state. In this case we regard the hybrid gates as entangling gates. Different methods exist for characterizing the enhancement of entanglement. In this section, we discuss entanglement enhancement by the hybrid SUM gate.

3.1. Entanglement measures for states and operators

There are various measures of entanglement for a normalized state $|\psi\rangle \in \mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$. Here, we shall use the von Neumann entropy

$$E(|\psi\rangle) = -\sum_{n=0}^{N_S-1} p_n \log p_n \tag{28}$$

where $\{p_n\}$ is defined in terms of the Schmidt decomposition of $|\psi\rangle$:

$$|\psi\rangle = \sum_{n=0}^{N_S-1} \sqrt{p_n}|\phi_n\rangle \otimes |\chi_n\rangle \quad p_n > 0 \quad \forall n \tag{29}$$

and log is always taken to be base 2. Definition (28) was adapted [23, 24] to define *operator entanglement*, as follows. Let $\mathcal{Q}$ be an operator acting on a hybrid space $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$, with the following Schmidt decomposition [24]:

$$\mathcal{Q} = \sum_{n=0}^{N_S-1} s_n A_n \otimes B_n \tag{30}$$

with $s_n > 0\, \forall n$, and the two operators A_n and B_n are orthonormal with respect to the Hilbert–Schmidt scalar product defined by $\langle A, B\rangle := \text{tr}(A^\dagger B)$ for A and B two arbitrary operators. In particular, $\|A\| := \sqrt{\text{tr}(A^\dagger A)}$ is the Hilbert–Schmidt norm of the operator A, and $\hat{A} := A/\|A\|$ if $\|A\| \neq 0$.

Since linear operators over a finite-dimensional vector space $\mathcal{H}_d$ can be regarded as d^2-dimensional vectors, we may think of $\widehat{\mathcal{Q}} \equiv \mathcal{Q}/\|\mathcal{Q}\|$ as a normalized state, which we denote by $|\widehat{\mathcal{Q}}\rangle\rangle$, so that (30) becomes

$$|\widehat{\mathcal{Q}}\rangle\rangle = \sum_{n=0}^{N_S-1} \sqrt{p_n}|A_n\rangle\rangle \otimes |B_n\rangle\rangle \tag{31}$$

where $\sqrt{p_n} = s_n/\|\mathcal{Q}\|$. In particular, if $\mathcal{Q}$ is unitary, then $\|\mathcal{Q}\| = \sqrt{d_c d_t}$. The operator entanglement [24]

$$E_{\mathrm{op}}(\mathcal{Q}) = -\sum_n \frac{s_n^2}{d_c d_t} \log\left(\frac{s_n^2}{d_c d_t}\right). \tag{32}$$

3.2. *Operator entanglement of the SUM gate*

In section 2, we essentially obtained in equation (13) the Schmidt decomposition of the operator $\mathcal{D}$ because the projection operators Π_s and the unitary operators X^s are mutually orthogonal, i.e.

$$\langle \Pi_r, \Pi_s \rangle = \|\Pi_s\|^2 \delta_{r,s} \qquad r, s \in \mathbb{Z}_{d_{\min}} \tag{33}$$

$$\langle X^r, X^s \rangle = \|X^s\|^2 \delta_{r,s} = d_t \delta_{r,s} \qquad r, s \in \mathbb{Z}_{d_t} \tag{34}$$

where we used $\|X^s\|^2 = \mathrm{tr}(X^{s\dagger} X^s) = \mathrm{tr}\, I = d_t$, because X^s is unitary. Hence, by dividing the operators Π_s and X^s in (13) by their norms, we immediately obtain the following Schmidt decomposition of $\mathcal{D}$:

$$\mathcal{D} := \sum_{s=0}^{d_{\min}-1} (\|\Pi_s\| \sqrt{d_t}) \widehat{\Pi}_s \otimes \widehat{X}^s \tag{35}$$

where for $d_c = K d_t + r$ we have

$$\|\Pi_s\| = \begin{cases} \sqrt{K+1} & (0 \leqslant s \leqslant r-1) \\ \sqrt{K} & (r \leqslant s \leqslant d_t - 1). \end{cases} \tag{36}$$

From equations (35) and (36), expression (31) yields immediately

$$E_{\mathrm{op}}(\mathcal{D}) = e_{\mathcal{D}}(d_c, d_t) \tag{37}$$

where (for $d_c = K d_t + r$)

$$e_{\mathcal{D}}(d_c, d_t) = -r \frac{K+1}{d_c} \log \frac{K+1}{d_c} - (d_t - r) \frac{K}{d_c} \log \frac{K}{d_c}. \tag{38}$$

Note that for $d_c < d_t$ the general expression (38) reduces simply to

$$e_{\mathcal{D}}(d_c, d_t) = \log d_c \qquad \text{for} \quad d_c < d_t \tag{39}$$

by substituting $K = 0$ and $r = d_c$.

3.3. *Entanglement produced by the SUM gate*

We prove the following lemma:

Lemma 1. *The entanglements generated by the hybrid SUM gate $\mathcal{D}$ on the following three initial product states (one without and two with ancillas)*

$$|\Psi_1\rangle \equiv |\gamma\rangle \otimes |t\rangle = \left(\frac{1}{\sqrt{d_c}} \sum_{m=0}^{d_c-1} |m\rangle\right) \otimes |t\rangle \tag{40}$$

$$|\Psi_2\rangle \equiv |\alpha\rangle \otimes |t\rangle = \left(\frac{1}{\sqrt{d_c}} \sum_{m=0}^{d_c-1} |m\rangle \otimes |m\rangle\right) \otimes |t\rangle \tag{41}$$

$$|\Psi_3\rangle \equiv |\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes \left(\frac{1}{\sqrt{d_t}} \sum_{n=0}^{d_t-1} |n\rangle \otimes |n\rangle \right) \tag{42}$$

where $|t\rangle$ is any of the computational states of the target space, are equal to the operator entanglement (38) of $\mathcal{D}$, i.e.

$$E(\mathcal{D}|\Psi_k\rangle) = E_{\mathrm{op}}(\mathcal{D}) = e_{\mathcal{D}}(d_c, d_t). \tag{43}$$

Proof. The three initial states have zero entanglement, since they were chosen to be *product* states. Therefore, the increase of entanglement due to $\mathcal{D}$ is equal to $E(\mathcal{D}|\Psi_k\rangle)$.

We shall now apply $\mathcal{D}$ to (40):

$$\left|\Psi_1^f\right\rangle \equiv \mathcal{D}|\gamma\rangle \otimes |t\rangle = \frac{1}{\sqrt{d_c}} \sum_{s=0}^{d_{\min}-1} \sum_{m=0}^{d_c-1} \Pi_s |m\rangle \otimes X^s |t\rangle. \tag{44}$$

Let $d_c = K d_t + r$ (note that $K = 0$ and $r = d_c$ if $d_c < d_t$). Hence,

$$\sum_{m=0}^{d_{\min}-1} \Pi_s |m\rangle = \begin{cases} |s\rangle + |s + d_t\rangle + \cdots + |s + K d_t\rangle = \sqrt{K+1}|\psi_s\rangle & \text{for} \quad 0 \leqslant s \leqslant r - 1 \\ |s\rangle + \cdots + |s + (K-1)d_t\rangle = \sqrt{K}|\psi_s\rangle & \text{for} \quad r \leqslant s \leqslant d_t - 1 \end{cases} \tag{45}$$

where the $|\psi_s\rangle$, $s \in \mathbb{Z}_{d_{\min}}$, are orthonormal states which, for $d_t < d_c$, span a d_t-dimensional *subspace* of $\mathcal{H}_{d_c}$. By substituting (45) into (44), we obtain the following Schmidt decomposition of the final state:

$$\left|\Psi_1^f\right\rangle = \mathcal{D}|\gamma\rangle \otimes |t\rangle = \sum_{s=0}^{d_c-1} \sqrt{p_s}|\psi_s\rangle \otimes |t + s\rangle \tag{46}$$

where

$$p_s = \begin{cases} (K+1)/d_c & \text{for} \quad 0 \leqslant s \leqslant r - 1 \\ K/d_c & \text{for} \quad r \leqslant s \leqslant d_t - 1. \end{cases} \tag{47}$$

By substituting the above equation into (29) we obtain exactly the same expression (38). Similarly, we can prove that the entanglement of $E(\mathcal{D}|\alpha\rangle \otimes |t\rangle)$ is also given by (38).

Finally, since the states $\{X^s|\beta\rangle\}$ are orthonormal for different s, we get essentially the same Schmidt decomposition for $\mathcal{D}|\alpha\rangle|\beta\rangle$ as in (46), and hence the same final entanglement. This result also follows from lemma 5 of [24]. □

The entanglement function (38) is plotted in figure 1. As the generated entanglement equals the operator entanglement according to equation (43), figure 1 presents E as the ordinate axis. We observe in figure 1 that the entanglement approaches $\log_2 d_t$ as d_c becomes large. We can see this asymptotic result in equation (38) by noting that

$$\frac{K+1}{d_c} = \frac{d_c + d_t - r}{d_c d_t} \to \frac{1}{d_t}$$

so the entanglement asymptotically approaches $\log d_t$ as observed in figure 1.

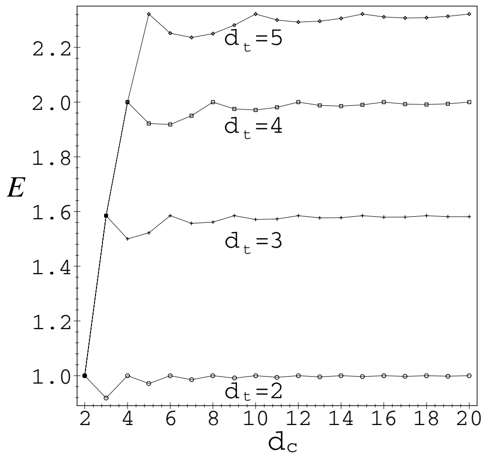

Figure 1. The operator entanglement of the hybrid SUM gate for different d_c and d_t.

4. Physical realization of hybrid gates

One can encode a qudit in physical systems such as spin systems and harmonic oscillators [2]. The Hilbert space associated with a spin-j system is spanned by the basis $\{|j, m\rangle; m = -j, \ldots, j\}$ and the $su(2)$ algebra is generated by $\{J_x, J_y, J_z\}$, with $[J_x, J_y] = \mathrm{i}J_z$ etc and $\left(J_x^2 + J_y^2 + J_z^2\right)|j, m\rangle = j(j+1)|j, m\rangle$. It is natural to define a number operator N and number states as follows:

$$N := J_z + jI \tag{48}$$

$$|n\rangle_j := |n - j\rangle \qquad (n = 0, \ldots, 2j). \tag{49}$$

Then we have $N|n\rangle_j = n|n\rangle_j$. In the spin system the operators X and Z are realized as

$$X = \sum_{n=0}^{2j} |n+1\rangle_{jj}\langle n| \tag{50}$$

$$Z = \exp[\mathrm{i}2\pi N/(2j+1)]. \tag{51}$$

4.1. Controlled-phase and SUM gates

We consider interaction between spin-j_1 and spin-j_2 systems, via the Hamiltonian $H = -gJ_{cz}J_{tz}$. Up to local unitary operators the evolution operator $\exp(\mathrm{i}tgJ_{cz}J_{tz})$ is equivalent to $U(t) = \exp(\mathrm{i}tgN_cN_t)$. By choosing $tg = 2\pi/(2j_t + 1) = 2\pi/d_t$, we obtain the unitary operator

$$V = \exp\left[\mathrm{i}\frac{2\pi}{d_t}N_cN_t\right] = \zeta_{d_t}^{N_cN_t} \tag{52}$$

which is just the controlled-phase gate [3]. On the other hand, we know that the SUM gate can be obtained from the controlled-phase gate as follows [25]:

$$\mathcal{D} = (I \otimes F^\dagger)\zeta_{d_t}^{N_cN_t}(I \otimes F). \tag{53}$$

Therefore, with the aid of F gate we realized the hybrid SUM gate.

4.2. Toffoli gate

Now let us see how to physically create a hybrid Toffoli gate. It is shown in [9, 26] that the interaction Hamiltonian $N_1N_2N_3$ (N_i correspond to spin-j_i and one j_i is equal to $1/2$) arises in ion-trap systems when coupling these operators N_i to a common continuous variable. The dimension of a spin-j_i system is given by $d_i = 2j_i + 1$. Therefore, we have the three-body controlled-phase gate

$$W(\theta) = \mathrm{e}^{\mathrm{i}\theta N_1N_2N_3}. \tag{54}$$

By choosing, say, $\theta = 2\pi/d_3$, we make $\mathcal{H}_{d_3}$ the target space while $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ becomes the control space. Then, by appending the appropriate F gate on the target system, we can realize the Toffoli gate acting on the systems $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2} \otimes \mathcal{H}_{d_3}$.

4.3. Fredkin gate

As a final remark we point out that we can construct a control-SWAP gate that acts on $\mathcal{H}_d \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty$ as a generalization of the controlled-SWAP gate acting on the $\mathcal{H}_2 \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty$ system [22].

The SWAP gate between two bosonic modes a_1 and a_2 is given by [22]

$$\mathcal{S}_{12} = \mathrm{e}^{\mathrm{i}\pi a_2^\dagger a_2}\, \mathrm{e}^{\frac{\pi}{2}(a_1^\dagger a_2 - a_2^\dagger a_1)}. \tag{55}$$

In an ion-trap system we can couple the spin-j system to two bosonic modes a_i ($i = 1, 2$) as [27, 28]

$$H_i = \chi N a_i^\dagger a_i. \tag{56}$$

Since operators H_i commute with each other, we can simulate the following Hamiltonian:

$$H = H_1 - H_2 = \chi N(a_1^\dagger a_1 - a_2^\dagger a_2) = 2\chi N J_z \tag{57}$$

where $J_z = \frac{1}{2}(a_1^\dagger a_1 - a_2^\dagger a_2)$. The operators J_z and $J_+ = a_1^\dagger a_2 = J_-^\dagger$ form the $su(2)$ Lie algebra. The evolution operator of the Hamiltonian H at time $t = -\pi/2\chi$ is given by

$$U = U(-\pi/2\chi) = \mathrm{e}^{\mathrm{i}\pi J_z N}. \tag{58}$$

The evolution operator U can be transformed to U' as

$$U' = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}J_x}\, U\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}J_x} = \mathrm{e}^{\mathrm{i}\pi J_y N} = \mathrm{e}^{\frac{\pi}{2}N(a_1^\dagger a_2 - a_2^\dagger a_1)} \tag{59}$$

where $J_x = (J_+ + J_-)/2$ and $J_y = (J_+ - J_-)/(2\mathrm{i})$. From equations (55), (56) and (59), we construct the controlled-SWAP gate (hybrid Fredkin gate) as

$$\mathcal{F} = \mathrm{e}^{\mathrm{i}\pi a_2^\dagger a_2 N}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{4}(a_1^\dagger a_2 + a_2^\dagger a_1)}\, \mathrm{e}^{\mathrm{i}\frac{\pi}{2}a_1^\dagger a_1 N}\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}a_2^\dagger a_2 N}\, \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}(a_1^\dagger a_2 + a_2^\dagger a_1)} = \mathcal{S}^N. \tag{60}$$

Therefore we have provided a controlled-SWAP gate on $\mathcal{H}_d \otimes \mathcal{H}_\infty \otimes \mathcal{H}_\infty$ systems in terms of five two-body operators.

5. Conjugation by the SUM gate

A conjugation by the SUM gate $\mathcal{D}$ is described by the following lemma:

Lemma 2. *The hybrid SUM gate $\mathcal{D}$ yields, by conjugation, an automorphism of the Pauli group $\mathcal{P}_{d_c} \otimes \mathcal{P}_{d_t}$, iff d_c/d_t is an integer K. More explicitly,*

$$\mathcal{D}(X \otimes I)\mathcal{D}^\dagger = X \otimes X \tag{61}$$

$$\mathcal{D}(I \otimes X)\mathcal{D}^\dagger = I \otimes X \tag{62}$$

$$\mathcal{D}(Z \otimes I)\mathcal{D}^\dagger = Z \otimes X \tag{63}$$

$$\mathcal{D}(I \otimes Z)\mathcal{D}^\dagger = \left(\sum_{s=0}^{d_c-1} \zeta_{d_c}^{-sd_c/d_t} P_s\right) \otimes Z \tag{64}$$

$$= Z^{-K} \otimes Z \qquad \text{for} \quad \frac{d_c}{d_t} = K. \tag{65}$$

Proof. By noting that

$$P_r X P_s = P_r|s+1\rangle\langle s| = |s+1\rangle\langle s|\delta_{r,s+1} \tag{66}$$

we obtain

$$\mathcal{D}(X \otimes X^k)\mathcal{D}^\dagger = \sum_{s=0}^{d_c-1} P_r X P_s \otimes X^{r+k-s} = X \otimes X^{k+1}. \tag{67}$$

This proves both (61) and (62) simultaneously. By noting that $Z^j = \sum_{s=0}^{d-1} \zeta_d^{sj} P_s$, we get

$$\mathcal{D}(Z \otimes I)\mathcal{D}^\dagger = \sum_{r,s,t=0}^{d_c-1} \zeta_{d_c}^s P_r P_s P_t \otimes X^{r-t} = Z \otimes I. \tag{68}$$

Finally, by using the commutation relation (6) and $\zeta_{d_t} = \left(\zeta_{d_c}\right)^{d_c/d_t}$, we obtain

$$\mathcal{D}(I \otimes Z)\mathcal{D}^\dagger = \sum_{s=0}^{d_c-1} P_s \otimes X^s Z X^{-s} = \sum_{s=0}^{d_c-1} P_s \otimes \zeta_{d_t}^{-s} Z = \sum_{s=0}^{d_c-1} \zeta_{d_c}^{-sd_c/d_t} P_s \otimes Z \tag{69}$$

$$= Z^{-K} \otimes Z \qquad \text{for} \quad d_c = K d_t. \tag{70}$$

□

Note that even if $d_c/d_t = K \geqslant 2$ is an integer, then $\mathcal{D}_{12}$ but not $\mathcal{D}_{21}$ will belong to the Clifford algebra of the hybrid Pauli group.

6. Summary

We considered quantum hybrid gates which act on tensor products of qudits of different dimensions. In particular, we constructed two-body hybrid SUM and partial-SWAP gates, and also many-body hybrid Toffoli and Fredkin gates. We have calculated the entanglement generated by the SUM gate. We describe a physical realization of these hybrid gates for spin systems. We also proved two lemmas, one related to entanglement generation with and without ancillas, and the other involving conjugation by the SUM gate.

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